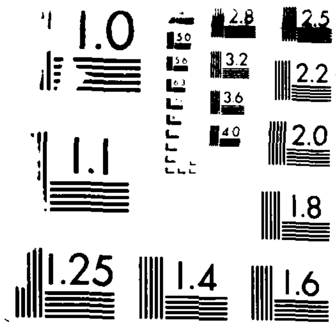
AD-A175 326 THE INTERNAL STATES OF THE 3-COMPONENT STANDBY SYSTEM 1/1
(U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA K E KEITEL
SEP 86

UNCLASSIFIED F/G 15/5 NL


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

THE INTERNAL STATES
OF THE
3-COMPONENT STANDBY SYSTEM

by

Karl-Heinz E. Keitel

September 1986

Thesis Advisor: James D. Esary

Approved for public release; distribution is unlimited.

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S)	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION Naval Postgraduate School	6b OFFICE SYMBOL (If applicable) Code 55
7a NAME OF MONITORING ORGANIZATION Naval Postgraduate School	
6c ADDRESS (City, State, and ZIP Code) Monterey, California 93943-5000	7b ADDRESS (City, State, and ZIP Code) Monterey, California 93943-5000
8a NAME OF FUNDING/SPONSORING ORGANIZATION	8b OFFICE SYMBOL (If applicable)
9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER	
8c ADDRESS (City, State, and ZIP Code)	10 SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO / PROJECT NO / TASK NO / WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)
THE INTERNAL STATES OF THE 3-COMPONENT STANDBY SYSTEM

12 PERSONAL AUTHOR(S)
Heinz, Karl-Heinz E.

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Master's Thesis	FROM ____ TO ____	1986 September	33

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	3- Component System, Conditional State Probabilities, Limiting Distribution, Failure Rates as a Linear Function of Conditional State Probabilities.

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

The study analyzes the internal states of a 3-component system with one active element and two spares in cold standby (pure replacement policy without repair). Elements of the system are assumed to have exponentially distributed lifetimes, however, special attention is paid to systems composed of components with different failure rates. The analysis is developed as a continuous-time Markovian process with stationary transition probabilities. Probabilities that exactly i components have failed by time t are calculated based on three levels of information: for systems in unknown condition, for systems known to be in UP-condition and for systems whose condition was not observed for some amount of time. A key part is the investigation of conditional probabilities of i components having failed by time t for a system known to be UP, the conditional limiting distribution as $t \to \infty$, and relation to the system failure rate. State probabilities for systems not monitored continuously for being UP are shown to be bounded between those corresponding to systems that are either observed constantly or not at all.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL James D. Esary	22b TELEPHONE (Include Area Code) (408) 646-2780 — 22c OFFICE SYMBOL Code 55Ey

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete — SECURITY CLASSIFICATION OF THIS PAGE

The Internal States of the 3-Component Standby System

by

Karl-Heinz E. Keitel
Captain, German Army
M.S. in Electrical Engineering
Fachhochschule des Heeres, Darmstadt, 1977

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1986

Author: Karl-Heinz E. Keitel

Approved by: James D. Esary, Thesis Advisor

Patricia A. Jacobs, Second Reader

Peter Purdue, Chairman,
Department of Operations Research

Kneale T. Marshall,
Dean of Information and Policy Sciences

ABSTRACT

The study analyzes the internal states of a 3-component system with one active element and two spares in cold standby (pure replacement policy without repair). Elements of the system are assumed to have exponentially distributed lifetimes, however, special attention is paid to systems composed of components with different failure rates. The analysis is developed as a continuous-time Markovian process with stationary transition probabilities. Probabilities that exactly i components have failed by time t are calculated based on three levels of information: for systems in unknown condition, for systems known to be in UP-condition, and for systems whose condition was not observed for some amount of time. A key part is the investigation of conditional probabilities of i components having failed by time t for a system known to be UP, the conditional limiting distribution as $t \to \infty$, and relation to the system failure rate. State probabilities for systems not monitored continuously for being UP are shown to be bounded between those corresponding to systems that are either observed constantly or not at all.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTION

This paper considers a non-repairable system of three components paying attention to various system structures that result from the use of non-identical components. The objective of the analysis is to derive and investigate state probabilities[1] for the system assuming different levels of information available about the system, and, as far as applicable, with respect to different arrangements of the components within the system. Although limited to a 3-component system, the basic results may be extended to non-homogeneous systems with more than three elements.

A stochastic model was found to be a convenient approach to the problem. It is developed in Chapter II together with the entire set of transition probabilities. That chapter will also show the system survival function, distribution function, and density function as functions that do not depend on the order of the components within the system. Chapter III will derive the state probabilities in three steps. A subset of the transition probabilities will immediatedly determine the state probabilities of a system in unknown condition. Special emphasis is given to a system which is known to be in working condition. In this case the resulting conditional state probabilities will be discussed in their limiting distribution and in their relation to the system failure rate; the position of the most reliable component in the system will turn out to be significant for the limiting conditional state probabilities. The final step will derive state probabilities for a system which is not monitored continuously.

Most computational work is not shown within the paper, however, the reader is provided with extensive tables which list the basic results for all possible system structures.

A. THE MODEL OF THE 3-COMPONENT STANDBY SYSTEM

The system under consideration throughout this paper consists of three components that may be thought of as an original and two spare components. Only one of the components is active at a given point in time and is exposed to failure. When a component fails it is replaced by one of the spares, if available. The

[1]The system will be said to be in state i if exactly i of the components are DOWN; thus the state probabilities correspond to the probability that 0,1, 2, or all components have failed at some time t (see Chapter II).

switch-over is assumed to cause no initial shock to the new component and to occur immediatedly after failure and in a negligible amount of time.

The system works (and is said to be UP) as long as at least one of its components is operable; it fails (is DOWN) if all components have failed.

No repair facility is provided for a broken component; thus it remains in the DOWN-condition throughout the mission once it has failed. This implies that the system itself will fail with certainty in a finite amount of time.

It will always be assumed that the system starts its useful life at $t=0$ with all components in the UP-condition.

B. IMPORTANT ASSUMPTIONS OF THE MODEL

The main characteristics of the system arise from three basic assumptions:

(a) Components in the spare status cannot fail. Therefore the failure rate of a spare is 0 (cold standby) and the lifetime of a component starts at the moment it is switched into the active state.

(b) The active component has an exponential life distribution, i.e., there is no effect of age on the component failure rate.

(c) All life lengths are mutually independent; i.e., the performance of the component currently active does not depend on the performance of its predecessor(s).

These assumptions are essential for the Markovian model used later on; they establish constant transition rates throughout the states of the system with changes possible only at the time a failure (= transition to the next state) occurs.

Note, however, that the life lengths of components in the active state are not assumed to be distributed identically. Thus the system may consist of components with different expected lifetimes.

Whenever it is required to distinguish between systems with respect to the homogeneity of their components, then:

(a) Structure 1 ($\lambda_0 \neq \lambda_1 \neq \lambda_2$) will refer to a system with three components each having a different failure rate (the most general case);

(b) Structure 2 (λ_i, $\lambda_j = \lambda_k = \lambda$) will denote a system with two components of the same kind (having failure rate λ) whereas the component used in the i^{th} place has a different failure rate λ_i;

(c) Structure 3 ($\lambda_0 = \lambda_1 = \lambda_2 = \lambda$) will denote a system of three identical components with common failure rate λ.

II. MODELING THE 3-COMPONENT STANDBY SYSTEM

The system may be found at any point of time t to be in one out of four possible states {0,1,2,3}. It will be said to be in state i if exactly i of the components are DOWN; thus the system will be UP in states {0,1,2} and DOWN in state 3. Since no repair is provided, it is immediate that the system after starting its life at $t=0$ in state 0 will transit into state 1 and 2 and finally be absorbed in the DOWN-state 3.

The amount of time, T_i, the system spends in state i before making the transition into state j is a random variable, distributed exponentially with mean $1/\lambda_i$, where λ_i is the failure rate of the component active in state i. By assumption T_i is independent of previous component failures in the system.

The system, or better yet the underlying process, may thus be modeled as a continuous-time Markovian process with stationary (homogeneous) transition probabilities [Refs. 1,2:pp. 26, 234].

Moreover, the process constitutes a pure birth process with a finite state space. Transitions are possible within the four states only from state i to state i+1 and occur at rate λ_i. State 3 is absorbing and allows no further transitions once it has been reached.

A. TRANSITION PROBABILITIES

By the memoryless property of the exponential sojourn times in a time-homogeneous Markovian process, the transition probabilities are conditioned only on the state of the system at time $t=0$ but not on the amount of time the system has already spent in that state.

Let $Z(t) = 0,1,2,3$ indicate the state of the system at time t, and

$$P_{ij}(t) = P[Z(t)=j \mid Z(0)=i]$$

be the probability that a process in state i at time $t=0$ will be in state j some time t later.

Then the transition probabilities $P_{ij}(t)$ may be obtained by solving the differential set of the Kolmogorov forwards equations for the pure birth process, as done by Ross [Ref. 2:p. 244].

In general formulation

$$P_{ii}(t) = e^{-\lambda_i t}$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} \; P_{i,j-1}(s)\, ds$$

for $i = 0,1,2$ and $i + 1 \leq j \leq 3$.

The computational work is too bulky to be presented within this paper, however, the resulting transition probabilities are completely listed in Tables III . . . VII (see Appendix A).

B. SYSTEM SURVIVAL-, DISTRIBUTION-, AND DENSITY FUNCTIONS

We will assume in what follows that at time $t = 0$ the system is new, that is that $Z(0) = 0$.

Let T be the failure time of the system. Then the probability that the system will survive a mission of length t is just the probability that the system will not transit into the DOWN-state before the end of the mission

$$\bar{F}(t) = P[T > t] = 1 - P_{03}(t)$$

Furthermore

$$F(t) = P[T \leq t] = P_{03}(t)$$

and

$$f(t) = d/dt\,[F(t)] = d/dt\,[P_{03}(t)]$$

The system survival function $\bar{F}(t)$, distribution function $F(t)$, and the density function $f(t)$ are summarized in Table VIII for the three possible structures of the system (see Appendix B). In either case, the formulas obtained for the functions are completely symmetric in the indices of the λ_i. Thus they do not depend on the order in which the various components are used in the system.

III. STATE PROBABILITIES IN THE 3-COMPONENT SYSTEM

In the following, state probabilities for the 3-component system are derived for three levels of information about the system:

(a) State probabilities $P_j(t)$ for a system which are unconditioned on any information about its condition after the initial start-up at $t=0$

(b) Conditional state probabilities $Q_j(t)$ for a system which is known to be in UP-condition at time t.

(c) Conditional state probabilities $P_j^*(t)$ for a system which was observed to be in UP-condition at some time $s<t$.

If the system was observed to be in a specific state i at some time after start-up then this information would reinitialize the Markovian process and either lead back to one of above cases or to a different model with less components. Therefore this case will not be discussed in the following analysis.

Most of the expressions developed for the state probabilities in this chapter contain several exponential terms (or a combination of two or more exponential and constant terms). Unless we restrict to simple cases, the attempt to calculate extreme points or intersection points for the functions will lead to equations which can only be solved using numerical procedures. Therefore, an approach is used which illustrates the functions in probability plots for various parameter values and stresses special features and limiting behaviour as $t \to \infty$.

A. STATE PROBABILITIES OF A SYSTEM IN UNKNOWN CONDITION

Within this section it is assumed that no information about the state or the condition of the system is available.

The probability that a system will be found in state j at time t, given it started its useful life at $t=0$ in state 0, is just the probability of a transition into state j during $(0,t)$. To simplify notation, we may define

$$P_j(t) \equiv P[Z(t)=j \mid Z(0)=0] = P_{0j}(t)$$

These state probabilities $P_j(t)$ have been plotted in Figures 3.1 to 3.3 for three failure rate combinations and over a standardized time axis (in multiples of the system $MTTF = 1/\lambda_0 + 1/\lambda_1 + 1/\lambda_2$).

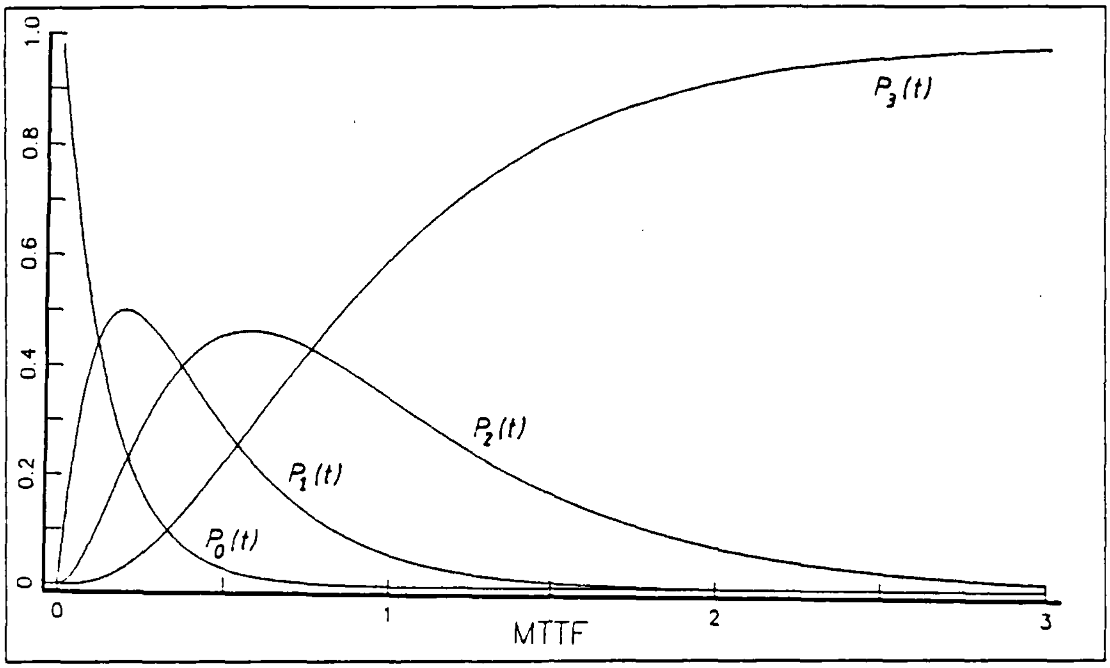


Figure 3.1 Unconditional State Probabilities $P_j(t)$, $\lambda_0:\lambda_1:\lambda_2$ = 4:2:1.

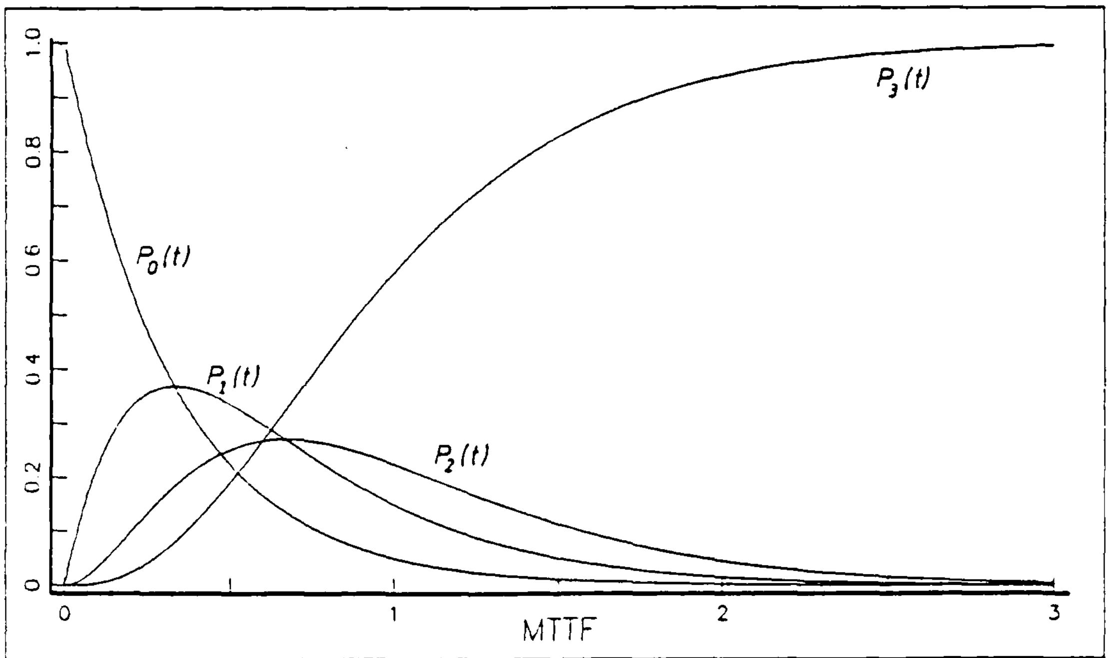


Figure 3.2 Unconditional State Probabilities $P_j(t)$, $\lambda_0:\lambda_1:\lambda_2$ = 2:2:2.

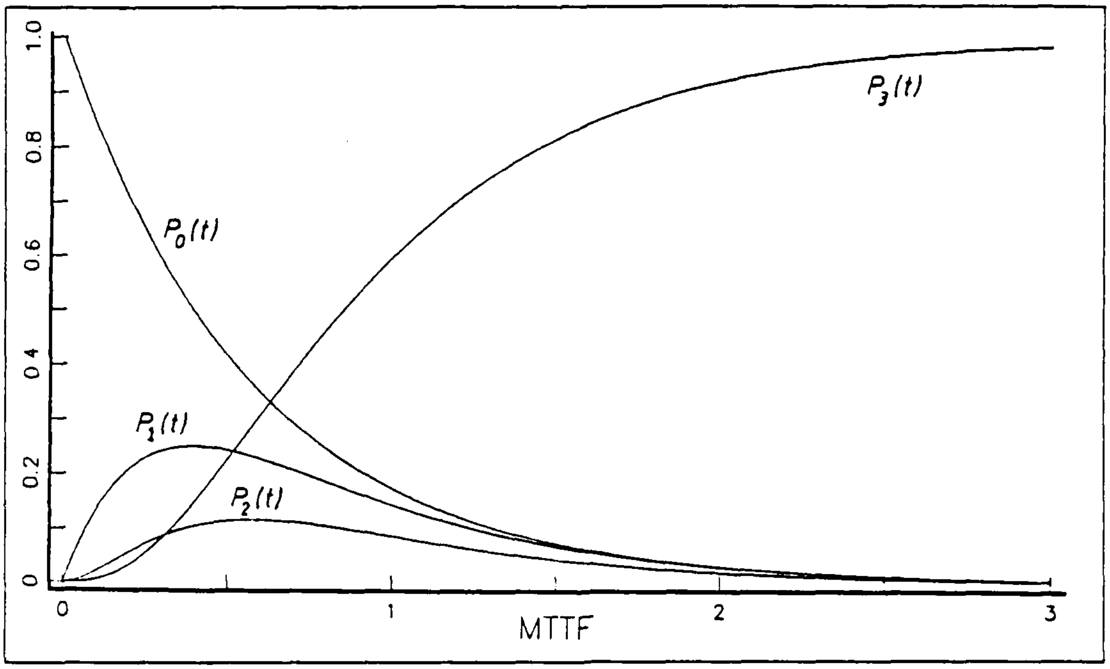


Figure 3.3 Unconditional State Probabilities $P_j(t)$, $\lambda_0:\lambda_1:\lambda_2 = 1:2:4$.

General information about the state probabilities $P_j(t)$ may be obtained by considering their first derivatives

$$d/dt\,[P_j(t)] = -\lambda_j P_j(t) + \lambda_{j-1} P_{j-1}(t)$$

The state probabilities $P_j(t)$ will increase if the probability of a transition into state j exceeds the probability of a transition out of state j at time t. Therefore it is immediate that $P_0(t)$ and $P_3(t)$ are monotonously decreasing, respectively increasing functions in t.

$P_1(t)$ and $P_2(t)$ appear to constitute unimodal functions[2] whose maxima can be located at

(a) For $P_1(t)$:

$$t = \frac{1}{\lambda_1 - \lambda_0}(\ln\lambda_1 - \ln\lambda_0) \qquad \text{if } \lambda_0 \neq \lambda_1$$

and

[2] A proof for this statement was not found in the literature, however, all trials to construct a counter example have failed within the work done for this study.

$$t = \frac{1}{\lambda} \qquad \text{if } \lambda_0 = \lambda_1 = \lambda$$

(b) For $P_2(t)$:

A closed form expression exists only for a homogeneous system:

$$t = \frac{2}{\lambda} \qquad \text{if } \lambda_0 = \lambda_1 = \lambda_2 = \lambda$$

The limiting state probabilities

$$P_j \equiv \lim_{t \to \infty} P_j(t) = \begin{cases} 0 & \text{if } j = 0,1,2 \\ 1 & \text{if } j = 3 \end{cases}$$

confirm the expectation that the system will finally be absorbed in the DOWN-state 3.

B. STATE PROBABILITIES FOR A SYSTEM IN KNOWN CONDITION

The following sections will consider systems which, without revealing their exact states, provide information enough to decide on their overall condition – system UP or DOWN – at time t.

If the system is known to be UP at time t then it may be in either one of the states {0,1,2} with probability

$$Q_j(t) \equiv P[Z(t) = j \mid \text{system UP}, Z(0) = 0] \qquad \text{(eqn 3.1)}$$

$$= \frac{P_{0j}(t)}{1 - P_{03}(t)} \qquad j = 0,1,2$$

which can be readily obtained from the transition probabilities in Tables III . . . VII.

Recalling that $P_{0j}(t) = P_j(t)$, the conditional state probabilities may be rewritten

$$Q_j(t) = \frac{1}{1 - P_3(t)} P_j(t) \qquad j = 0,1,2$$

and, for a fixed t, can thus be obtained by multiplying the state probabilities $P_j(t)$, we would get without the additional information, each by the same factor $c = [1 - P_3(t)]^{-1}$. This factor c is greater than or equal to 1 for all t and, as a monotonously increasing function of t, may get as large as infinity as $t \to \infty$.

The probability plots for the conditional state probabilities $Q_j(t)$, in Figures 3.4 to 3.6, are based on the same ratios in the failure rates as the plots of Figures 3.1 to 3.3 and may provide a rough idea about the shape of the functions $Q_j(t)$.

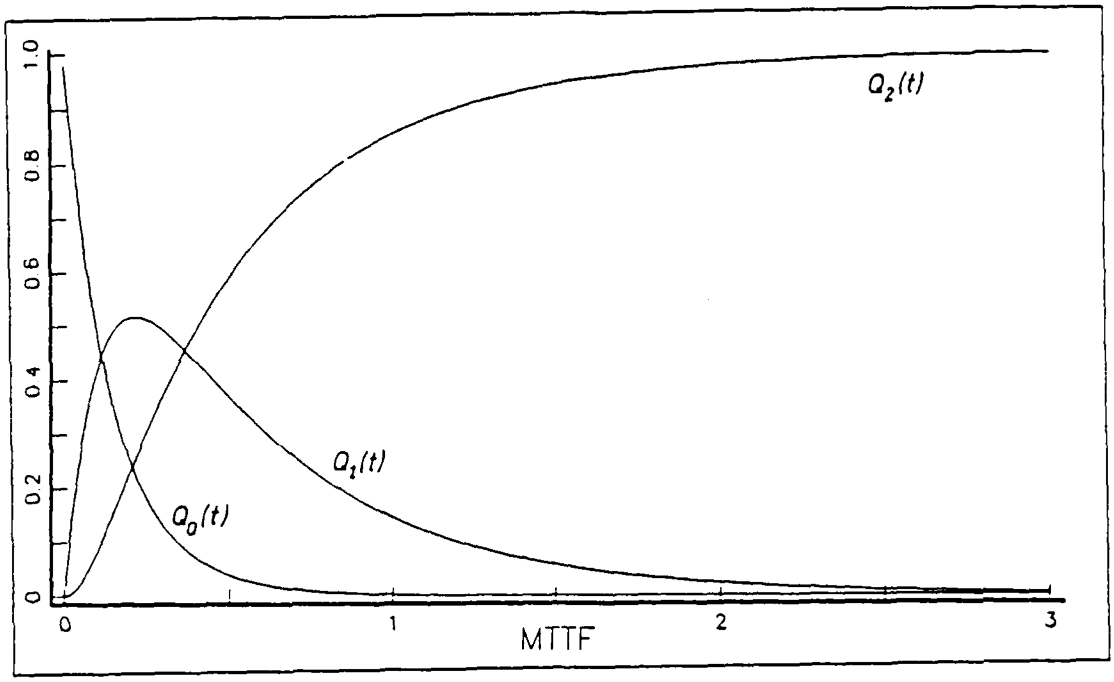


Figure 3.4 Conditional State Probabilities $Q_j(t)$, $\lambda_0:\lambda_1:\lambda_2 = 4:2:1$.

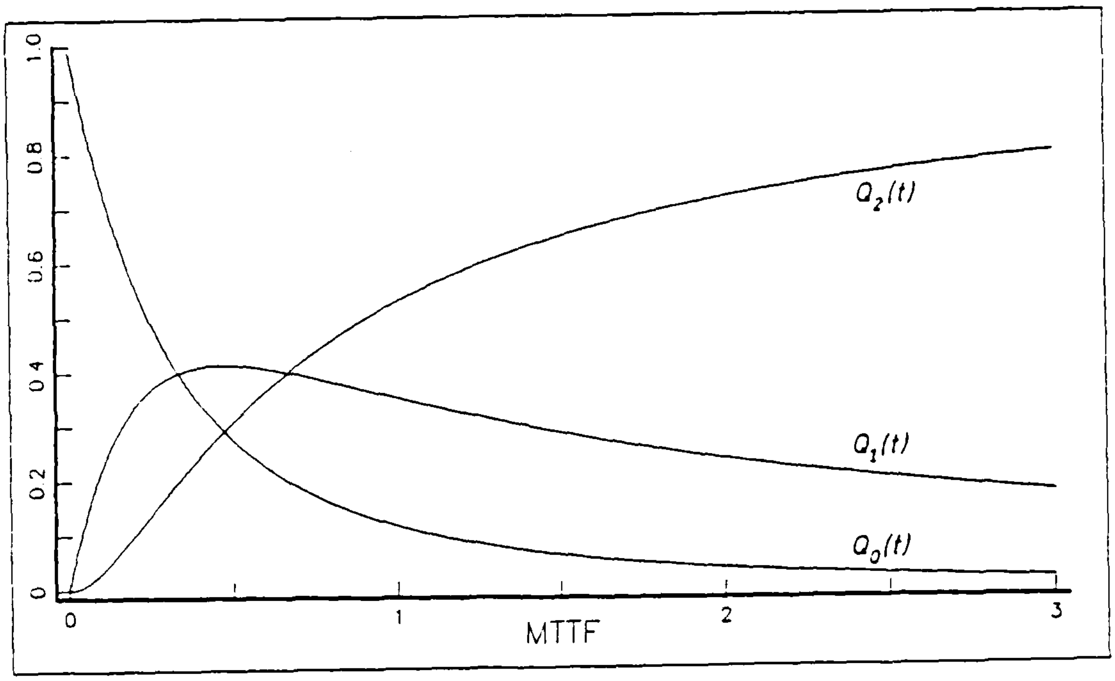


Figure 3.5 Conditional State Probabilities $Q_j(t)$, $\lambda_0:\lambda_1:\lambda_2 = 2:2:2$.

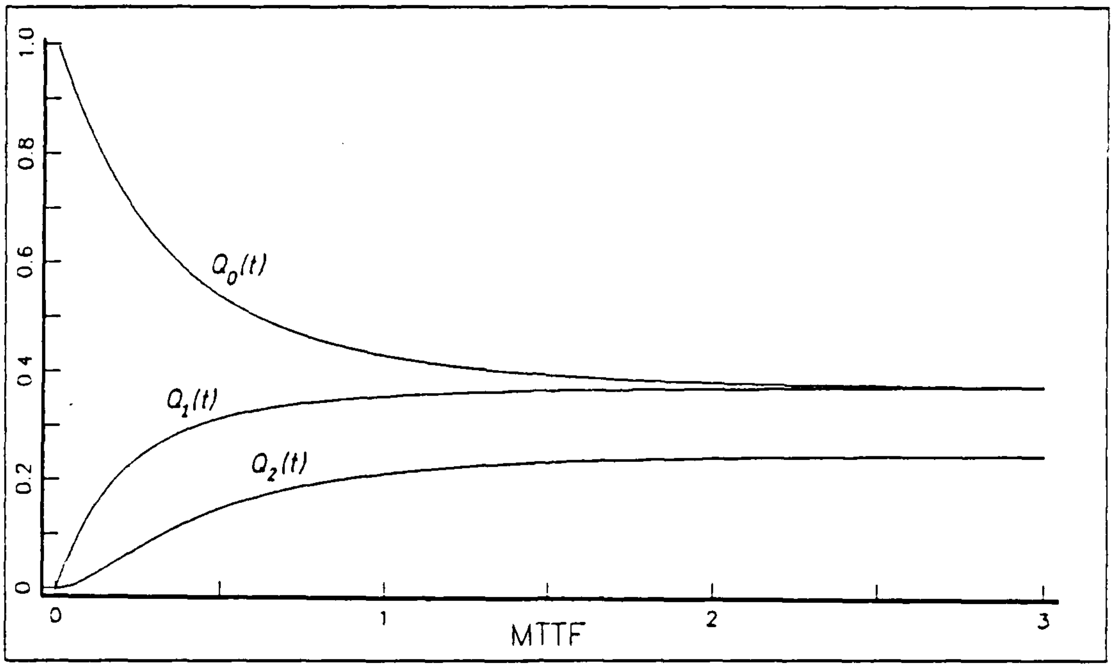


Figure 3.6 Conditional State Probabilities $Q_j(t)$, $\lambda_0:\lambda_1:\lambda_2 = 1:2:4$.

1. **Limiting State Probabilities**

The plot of Figure 3.6 suggests the existence of limiting conditional state probabilities that are not necessarily equal to 0 or 1, unlike the case of the unconditional state probabilities $P_j(t)$. State 2 does not seem to take over the part of an absorbing state in all cases, as we might have expected.

To derive the limiting conditional state probabilities

$$Q_j \equiv \lim_{t\to\infty} Q_j(t) = \lim_{t\to\infty} P[Z(t)=j \mid \text{system UP}, Z(0)=0]$$

for $j=0,1,2$ it is necessary to pay close attention to the order in which components with different failure rates are used in the system. We also need to specify which component has the minimum failure rate. It is therefore convenient to derive the limiting probabilities separately for the different orderings of component failure rates possible within the system.

a. Structure 1 $(\lambda_0 \neq \lambda_1 \neq \lambda_2)$

In this structure the failure rates of the components are all different; we can, therefore, specify a unique minimum failure rate λ_i. It should be recalled that the

indices of the λ_i denote the position of the corresponding component within the system (initial component or i^{th} spare); thus e.g., $\lambda_1 = \min(\lambda_0, \lambda_1, \lambda_2)$ will express that the first spare is assumed to be the component with the smallest failure rate used in the system.

Using Equation 3.1 and Table III we get the limiting probability matrix shown in Table I.

TABLE I
LIMITING STATE PROBABILITIES Q_j, $\lambda_0 \neq \lambda_1 \neq \lambda_2$

	Q_0	Q_1	Q_2
$\lambda_0 = \min(\lambda_0, \lambda_1, \lambda_2)$	$\frac{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)}{\lambda_1 \lambda_2}$	$\frac{\lambda_0(\lambda_2 - \lambda_0)}{\lambda_1 \lambda_2}$	$\frac{\lambda_0}{\lambda_2}$
$\lambda_1 = \min(\lambda_0, \lambda_1, \lambda_2)$	0	$\frac{\lambda_2 - \lambda_1}{\lambda_2}$	$\frac{\lambda_1}{\lambda_2}$
$\lambda_2 = \min(\lambda_0, \lambda_1, \lambda_2)$	0	0	1

b. Structure 2 $(\lambda_i, \lambda_j = \lambda_k = \lambda)$

The limiting state probabilities Q_j may be achieved either by calculating the limits of the $Q_j(t)$, as $t \rightarrow \infty$, or they may be derived from the general structure (Table I) using the following rules:

(a) If $\lambda_i < \lambda$, i.e., if there is a unique component associated with the minimum failure rate, then use that row of the matrix which corresponds to the position of the singular component within the system

(b) If $\lambda < \lambda_i$, i.e., if the minimum failure rates applies to two components, then use that row of the matrix which corresponds to the position of that component with failure rate λ which is used last within the system (i.e., if two rows in the matrix apply then use the lower one).

The limiting state probabilities are listed in Table II.

TABLE II
LIMITING STATE PROBABILITIES Q_j, $\lambda_i, \lambda_j = \lambda_k = \lambda$

System Structure		Q_0	Q_1	Q_2
Case λ_0-λ-λ:	$\lambda_0 < \lambda$	$\frac{(\lambda_0-\lambda)^2}{\lambda^2}$	$\frac{\lambda_0(\lambda-\lambda_0)}{\lambda^2}$	$\frac{\lambda_0}{\lambda}$
	$\lambda_0 > \lambda$	0	0	1
Case λ-λ_1-λ:	$\lambda_1 < \lambda$	0	$1 - \frac{\lambda_1}{\lambda}$	$\frac{\lambda_1}{\lambda}$
	$\lambda_1 > \lambda$	0	0	1
Case λ-λ-λ_2:	$\lambda_2 < \lambda$	0	0	1
	$\lambda_2 > \lambda$	0	$1 - \frac{\lambda}{\lambda_2}$	$\frac{\lambda}{\lambda_2}$

c. *Structure 3* ($\lambda_0 = \lambda_1 = \lambda_2 = \lambda$)

In this structure the system consists of three identical components. The limiting state probabilities may be calculated to be

$Q_0 = 0$

$Q_1 = 0$

$Q_2 = 1$

or they may be looked up in Table I by taking the last row of the matrix (in accordance to the rule that points to the last used component, if more than one has the same minimum failure rate).

2. **Interpretation of the Limiting Conditional State Probabilities**

One might have expected to find systems working on their very last spare, if they are still alive after use for a sufficiently large amount of time. However, the results show, that under certain arrangements of the component failure rates, we may very well find systems working even on their initial component.

An explanation for this system behaviour is most readily obtained by looking at the upper triangular probability matrix in Table I for a system composed of components all with different failure rates. Obviously, the component with the minimum failure rate takes on an important position within the system. Given that the system is UP, the limiting conditional state distribution will put all its probability mass onto states corresponding to the most reliable component (main diagonal in the matrix) and its possible replacements. Thus functioning systems drop through initial less reliable components at least onto their most reliable ones. One will always find systems which remain working on their most reliable component. Others will continue to transit into further states (unless the most reliable component is also the last one available). If the initial component is also the most reliable one then either one of the spares may be found to carry the system in the limiting distribution; there is no second 'drop through' to a spare with a smaller failure rate. The probabilities that a system will eventually be found in states corresponding to the most reliable component and its possible replacements depend only on the failure rates associated with these components.

In case of the more restrictive structures 2 and 3 we are not always able to define a unique component to be the most reliable one. Previous observations, however, remain valid if we assign the attribute 'most reliable' to that component among identical ones, which is used last within the system. Therefore, if more than one component shares the same minimum failure rate, we will observe systems dropping through identical components and, in the limiting probability, a system with three identical components will be found working on its last spare with certainty.

3. System Failure Rate as a Linear Function of the Conditional State Probabilities

The system failure rate r(t) may be considered to be the probability of an instantaneous system failure at any time t, given the system survived up to time t. Then

$$
\begin{aligned}
r(t) &= \lim_{h\to 0} \frac{P[\text{system fails in } (t,t+h) \mid \text{system UP at } t]}{h} \\
&= \lim_{h\to 0} \sum_{i=0}^{2} \frac{P[Z(t+h)=3 \mid Z(t)=i]\, P[Z(t)=i \mid \text{system UP}]}{h} \\
&= \lim_{h\to 0} \sum_{i=0}^{2} \frac{P_{i3}(h)\, Q_i(t)}{h} \\
&= \sum_{i=0}^{2} Q_i(t) \lim_{h\to 0} \frac{P_{i3}(h)}{h}
\end{aligned}
$$

Since the transition probabilities

$$p_{ij} \equiv \lim_{h\to 0} \frac{P_{ij}(h)}{h}$$

are constant for a Markovian process with stationary transition probabilities, the system failure rate can be expressed as a linear combination of the conditional state probabilities $Q_j(t)$.

For the 3-component system under consideration, we have $p_{03} = p_{13} = 0$ and $p_{23} = \lambda_2$. The system failure rate thus reduces to

$$r(t) = \lambda_2 Q_2(t)$$

and calculates to:

(a) Structure 1

$$r(t) = \frac{\lambda_0\lambda_1\lambda_2\,[(\lambda_1-\lambda_2)e^{-\lambda_0 t} - (\lambda_0-\lambda_2)e^{-\lambda_1 t} + (\lambda_0-\lambda_1)e^{-\lambda_2 t}]}{\lambda_1\lambda_2(\lambda_1-\lambda_2)e^{-\lambda_0 t} - \lambda_0\lambda_2(\lambda_0-\lambda_2)e^{-\lambda_1 t} + \lambda_0\lambda_1(\lambda_0-\lambda_1)e^{-\lambda_2 t}}$$

(b) Structure 2

$$r(t) = \lambda_i\lambda^2 \frac{[(\lambda_i-\lambda)t - 1]e^{-\lambda t} + e^{-\lambda_i t}}{[\lambda_i(\lambda_i-2\lambda)+\lambda\lambda_i(\lambda_i-\lambda)t]e^{-\lambda t} + \lambda^2 e^{-\lambda_i t}}$$

(c) Structure 3

$$r(t) = \frac{\frac{1}{2}\lambda^3 t^2}{1 + \lambda t + \frac{1}{2}\lambda^2 t^2}$$

These system failure rates are continuous functions and, as shown in general by Barlow and Proschan [Ref. 3:p. 100], strictly increasing over t. Thus the 3-component system wears out as it ages.

The system failure rate r(t) does not depend on the order in which the components are arranged within the system (however, $Q_j(t)$ does). In the limit, as $t \to \infty$, r(t) always approaches the minimum failure rate found in the system.

C. STATE PROBABILITIES FOUND IN A SYSTEM NOT OBSERVED CONTINUOUSLY

In the previous sections state probabilities were derived for two special cases:

(a) State probabilities $P_j(t)$ for a system that did not reveal any information about its condition at time t.

(b) Conditional state probabilities $Q_j(t)$ for a system that was known to be in UP-condition at time t.

We now want to combine both cases and calculate state probabilities $P_j^*(t)$ for a system that was observed in UP-condition at some time s in the past but whose present condition (at time t) is unknown. Since the system may have failed during (s,t) the DOWN-state 3 must be included in the analysis.

As before, we implicitly assume that the system was new when started up at t=0, i.e., that Z(0)=0. By conditioning

$$\begin{aligned} P_j^*(t) &\equiv P[Z(t)=j \mid \text{system UP at } s] \\ &= \sum_{i \le j} P[Z(t)=j \mid Z(s)=i]\, P[Z(s)=i \mid \text{sytem UP at } s] \\ &= \sum_{i \le j} P_{ij}(t\text{-}s)\, Q_i(s) \end{aligned}$$

and for j=0,1,2 by Equation 3.1

$$P_j^*(t) = \sum_{i \le j} P_{ij}(t\text{-}s) \frac{P_{0i}(s)}{1 - P_{03}(s)} \qquad j=0,1,2$$

Note: This equation is also valid for $P_3^*(t)$, if the sum is taken for $i<j$ only (since $P[Z(s)=3 \mid \text{system UP at } s]=0$).

After calculating the sum, above equation reduces to

$$P_j^*(t) = \frac{1}{1 - P_{03}(s)} P_{0j}(t) \qquad j=0,1,2$$

or, in terms of state probabilities $P_j(t)$,

$$P_j^*(t) = \frac{1}{1 - P_3(s)} P_j(t) \qquad j=0,1,2$$

Again, the set of conditional state probabilities for the UP-states of the system may be obtained by multiplying each of the state probabilities $P_j(t)$ by some factor $c^* = [1 - P_3(s)]^{-1}$. However, c^* is a function of s only and, once s is fixed, it remains constant over $t \ge s$.

The time s, at which the system was observed to be UP, can be located anywhere in the interval (0,t). If s=0 then $c^* = 1$, and $P_j^*(t)$ reduces to $P_j(t)$, as expected (the implicit assumption Z(0)=0 does already include the 'system UP' information). If s=t then $c^* = c$, and we obtain the conditional state probabilities $Q_j(t)$ investigated in the last section.

Since $P_3(t)$ is monotonously increasing over t, c^* is always found to be $1 \leq c^* \leq c$. Therefore the conditional state probabilities $P_j^*(t)$ are bounded by $P_j(t)$ and $Q_j(t)$ such that $P_j(t) \leq P_j^*(t) \leq Q_j(t)$, $j = 0,1,2$.

Switching between $P_j(t)$, $P_j^*(t)$, and $Q_j(t)$ enables us to calculate the state probabilities for different techniques used in observing the condition of the system. If the condition of a system is monitored continuously and found to be UP over a period of time then the state probabilities calculated at any time within the interval will follow the $Q_j(t)$ curves. As soon as observations are interrupted the state probabilities are determined by $P_j^*(t)$. If no further observation is taken from the system, then the increasing uncertainty about the condition of the system will force the state probabilities associated with the UP-states asymptotically towards their lower boundaries $P_j(t)$. At the same time, the DOWN-state probability

$$P_3^*(t) = \frac{1}{1 - P_3(s)} [P_3(t) - P_3(s)]$$

increases from 0 towards its upper bound $P_3(t)$.

This is illustrated in Figure 3.7 for state 1 of the system. It is assumed that the system was initially monitored continuously (in UP-condition). At time $s_1 = 1 \mathrm{xMTTF}$ observation was discontinued until, in a single observation at $s_2 = 2 \mathrm{xMTTF}$, the system was found to be still in UP-condition. The probability plot for state 1 (bold line) is therefore composed of three partial plots using $Q_1(t)$, for $0 \leq t \leq s_1$, and $P_1^*(t)$ with $s = s_1$ (for $s_1 \leq t \leq s_2$) and $s = s_2$ (for $t \geq s_2$).

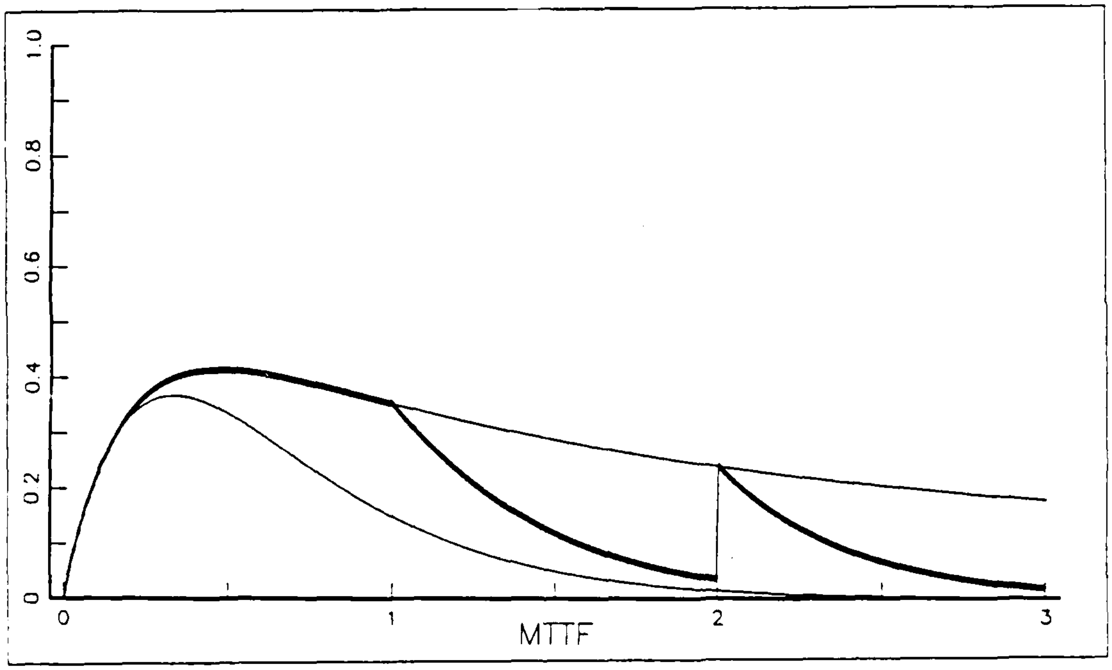


Figure 3.7 Composed State Probabilities for State 1, $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$.

APPENDIX A

TRANSITION PROBABILITIES FOR THE 4-STATE MODEL

TABLE III

TRANSITION PROBABILITIES STRUCTURE 1 ($\lambda_0 \neq \lambda_1 \neq \lambda_2$)

$$P_{00}(t) = e^{-\lambda_0 t}$$

$$P_{01}(t) = \frac{\lambda_0}{\lambda_0 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_0 t}\right)$$

$$P_{02}(t) = \frac{\lambda_0 \lambda_1}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}\left[(\lambda_1 - \lambda_2)e^{-\lambda_0 t} - (\lambda_0 - \lambda_2)e^{-\lambda_1 t} + (\lambda_0 - \lambda_1)e^{-\lambda_2 t}\right]$$

$$P_{03}(t) = 1 - \frac{\lambda_0 \lambda_1 \lambda_2}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}\left[\frac{\lambda_1 - \lambda_2}{\lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0 - \lambda_2}{\lambda_1} e^{-\lambda_1 t} + \frac{\lambda_0 - \lambda_1}{\lambda_2} e^{-\lambda_2 t}\right]$$

$$P_{11}(t) = e^{-\lambda_1 t}$$

$$P_{12}(t) = \frac{\lambda_1}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

$$P_{13}(t) = 1 - \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_1 - \lambda_2}$$

$$P_{22}(t) = e^{-\lambda_2 t}$$

$$P_{23}(t) = 1 - e^{-\lambda_2 t}$$

$$P_{33}(t) = 1$$

TABLE IV

TRANSITION PROBABILITIES STRUCTURE 2A (λ_0, $\lambda_1 = \lambda_2 = \lambda$)

$$P_{00}(t) = e^{-\lambda_0 t}$$

$$P_{01}(t) = \frac{\lambda_0}{\lambda_0 - \lambda}\left(e^{-\lambda t} - e^{-\lambda_0 t}\right)$$

$$P_{02}(t) = \frac{\lambda_0 \lambda}{(\lambda_0 - \lambda)^2}\left\{[(\lambda_0 - \lambda)t - 1]e^{-\lambda t} + e^{-\lambda_0 t}\right\}$$

$$P_{03}(t) = 1 - \frac{[\lambda_0(\lambda_0 - 2\lambda) + \lambda\lambda_0(\lambda_0 - \lambda)t]e^{-\lambda t} + \lambda^2 e^{-\lambda_0 t}}{(\lambda_0 - \lambda)^2}$$

$$P_{11}(t) = e^{-\lambda t}$$

$$P_{12}(t) = \lambda t e^{-\lambda t}$$

$$P_{13}(t) = 1 - (1 + \lambda t)e^{-\lambda t}$$

$$P_{22}(t) = e^{-\lambda t}$$

$$P_{23}(t) = 1 - e^{-\lambda t}$$

$$P_{33}(t) = 1$$

TABLE V

TRANSITION PROBABILITIES STRUCTURE 2B ($\lambda_0 = \lambda_2 = \lambda, \lambda_1$)

$$P_{00}(t) = e^{-\lambda t}$$

$$P_{01}(t) = \frac{\lambda}{\lambda_1 - \lambda}\left(e^{-\lambda t} - e^{-\lambda_1 t}\right)$$

$$P_{02}(t) = \frac{\lambda\lambda_1}{\lambda_1 - \lambda}\left\{[(\lambda_1 - \lambda)t - 1]e^{-\lambda t} + e^{-\lambda_1 t}\right\}$$

$$P_{03}(t) = 1 - \frac{[\lambda_1(\lambda_1 - 2\lambda) + \lambda\lambda_1(\lambda_1 - \lambda)t]e^{-\lambda t} + \lambda^2 e^{-\lambda_1 t}}{\lambda_1 - \lambda}$$

$$P_{11}(t) = e^{-\lambda_1 t}$$

$$P_{12}(t) = \frac{\lambda_1}{\lambda_1 - \lambda}\left(e^{-\lambda t} - e^{-\lambda_1 t}\right)$$

$$P_{13}(t) = 1 - \frac{\lambda_1 e^{-\lambda t} - \lambda e^{-\lambda_1 t}}{\lambda_1 - \lambda}$$

$$P_{22}(t) = e^{-\lambda t}$$

$$P_{23}(t) = 1 - e^{-\lambda t}$$

$$P_{33}(t) = 1$$

TABLE VI

TRANSITION PROBABILITIES STRUCTURE 2C ($\lambda_0 = \lambda_1 = \lambda, \lambda_2$)

$$P_{00}(t) = e^{-\lambda t}$$

$$P_{01}(t) = \lambda t e^{-\lambda t}$$

$$P_{02}(t) = \frac{\lambda^2}{(\lambda_2-\lambda)^2}\{[(\lambda_2-\lambda)t - 1]e^{-\lambda t} + e^{-\lambda_2 t}\}$$

$$P_{03}(t) = 1 - \frac{[\lambda_2(\lambda_2-2\lambda) + \lambda\lambda_2(\lambda_2-\lambda)t]e^{-\lambda t} + \lambda^2 e^{-\lambda_2 t}}{(\lambda_2-\lambda)^2}$$

$$P_{11}(t) = e^{-\lambda t}$$

$$P_{12}(t) = \frac{\lambda}{\lambda_2-\lambda}(e^{-\lambda t} - e^{-\lambda_2 t})$$

$$P_{13}(t) = 1 - \frac{\lambda_2 e^{-\lambda t} - \lambda e^{-\lambda_2 t}}{\lambda_2-\lambda}$$

$$P_{22}(t) = e^{-\lambda_2 t}$$

$$P_{23}(t) = 1 - e^{-\lambda_2 t}$$

$$P_{33}(t) = 1$$

TABLE VII

TRANSITION PROBABILITIES STRUCTURE 3 ($\lambda_0 = \lambda_1 = \lambda_2 = \lambda$)

$$P_{00}(t) = e^{-\lambda t}$$

$$P_{01}(t) = \lambda t e^{-\lambda t}$$

$$P_{02}(t) = \tfrac{1}{2}\lambda^2 t^2 e^{-\lambda t}$$

$$P_{03}(t) = 1 - (1 + \lambda t + \tfrac{1}{2}\lambda^2 t^2)e^{-\lambda t}$$

$$P_{11}(t) = e^{-\lambda t}$$

$$P_{12}(t) = \lambda t e^{-\lambda t}$$

$$P_{13}(t) = 1 - (1 + \lambda t)e^{-\lambda t}$$

$$P_{22}(t) = e^{-\lambda t}$$

$$P_{23}(t) = 1 - e^{-\lambda t}$$

$$P_{33}(t) = 1$$

APPENDIX B
SYSTEM CHARACTERISTICS

TABLE VIII

SYSTEM SURVIVAL-, DISTRIBUTION-, DENSITY FUNCTIONS

Structure 1 ($\lambda_0 \neq \lambda_1 \neq \lambda_2$):

$$\bar{F}(t) = \frac{\lambda_0\lambda_1\lambda_2}{(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}\left[\frac{\lambda_1-\lambda_2}{\lambda_0}e^{-\lambda_0 t} - \frac{\lambda_0-\lambda_2}{\lambda_1}e^{-\lambda_1 t} + \frac{\lambda_0-\lambda_1}{\lambda_2}e^{-\lambda_2 t}\right]$$

$$F(t) = 1 - \frac{\lambda_0\lambda_1\lambda_2}{(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}\left[\frac{\lambda_1-\lambda_2}{\lambda_0}e^{-\lambda_0 t} - \frac{\lambda_0-\lambda_2}{\lambda_1}e^{-\lambda_1 t} + \frac{\lambda_0-\lambda_1}{\lambda_2}e^{-\lambda_2 t}\right]$$

$$f(t) = \frac{\lambda_0\lambda_1\lambda_2}{(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}\left[(\lambda_1-\lambda_2)e^{-\lambda_0 t} - (\lambda_0-\lambda_2)e^{-\lambda_1 t} + (\lambda_0-\lambda_1)e^{-\lambda_2 t}\right]$$

Structure 2 ($\lambda_i, \lambda_j = \lambda_k = \lambda$):

$$\bar{F}(t) = \frac{[\lambda_i(\lambda_i-2\lambda) + \lambda\lambda_i(\lambda_i-\lambda)t]e^{-\lambda t} + \lambda^2 e^{-\lambda_i t}}{(\lambda_i-\lambda)^2}$$

$$F(t) = 1 - \frac{[\lambda_i(\lambda_i-2\lambda) + \lambda\lambda_i(\lambda_i-\lambda)t]e^{-\lambda t} + \lambda^2 e^{-\lambda_i t}}{(\lambda_i-\lambda)^2}$$

$$f(t) = \frac{\lambda_i\lambda^2}{(\lambda_i-\lambda)^2}\{[(\lambda_i-\lambda)t - 1]e^{-\lambda t} + e^{-\lambda_i t}\}$$

Structure 3 ($\lambda_0 = \lambda_1 = \lambda_2 = \lambda$):

$$\bar{F}(t) = (1 + \lambda t + \tfrac{1}{2}\lambda^2 t^2)e^{-\lambda t}$$

$$F(t) = 1 - (1 + \lambda t + \tfrac{1}{2}\lambda^2 t^2)e^{-\lambda t}$$

$$f(t) = \tfrac{1}{2}\lambda^3 t^2 e^{-\lambda t}$$

LIST OF REFERENCES

1. Birolini, Alessandro, *On the Use of Stochastic Processes in Modeling Reliability Problems*, Springer-Verlag Berlin Heidelberg, 1985.

2. Ross, S.M., *Introduction to Probability Models*, 3rd ed., Academic Press, Inc., 1985.

3. Barlow, R.E. and Proschan, F., *Statistical Theory of Reliabilty and Life Testing*, McArdle Press, Inc., Silver Spring, MD, 1981.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Technical Information Center Cameron Station Alexandria, Virginia 22304-6145	2
2.	Library, Code 0142 Naval Postgraduate School Monterey, California 93943-5002	2
3.	Department Chairman, Code 55Pd Department of Operations Research Naval Postgraduate School Monterey, California 93943-5000	1
4.	Professor James D. Esary Code 55Ey Naval Postgraduate School Monterey, California 93943-5000	2
5.	Professor Patricia A. Jacobs Code 55Jc Naval Postgraduate School Monterey, California 93943-5000	1
6.	Captain Karl-Heinz Keitel 4./Instandsetzungsbatallion 3 Von Goeben-Kaserne 2160 Stade, West Germany	2

END

2-81

DTIC